Topological quantum field theory and crossing number

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Abstract

In this paper, we construct a new topological quantum field theory of cohomological type and show that its partition function is a crossing number.

I. Introduction and Notation

Topological field theories(TFT), first suggested by M.Atiya[4] based on the work of A.Fleer[5], where largely introduced by E. Witten in [1], [3]. It may be grouped into two classes: "Schwarz type" and "cohomological type". In this paper, we will focus on topological field theories of cohomological type. Cohomological field theory has many important applications in mathematics and physics. However, the true relevance of topological theories to more traditional problems in quantum field therory remains to be an open question. It is a very instrsting field for many famous mathematicians and physicsts.

Topological field theories of cohomological type describe intersection theory in moduli spaces in the language of local quantum field theory. The moduli spaces that arise in physics are very canonical and fundamental objects. In[3] Witten introduced a Lagrangian leading to a topological quantum field theory in which

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the Donaldson invariants of 4-manifolds appear as expectation values of the observables.

It is well know that many topological invariants can be interpreted in terms of topological field theories. As an important topological invariant, crossing number plays an important role in defining Seiberg-Witten invariants [2].

In this paper, we first constuct a topological quantum field theory in terms of some real fields and some complex fields, then develop the path integration of complex fields. Finally, we show that its partition function is the crossing number.

Now we give some notation conventions and some conclusition that will be showed in latter sections.

Let E and F be two complex vector bundles over a smooth compact oriented manifold with real dimension $\dim M = 2n$. Denote by $E \ominus F$ the equivalent class of a pair (E, F) under the equivalence relation $(E_1, F_1) \sim (E_2, F_2) \Leftrightarrow E_1 \oplus F_2 \oplus C^N \cong E_2 \oplus F_1 \oplus C^N$ for some integer N. Let D be section $D \in C^\infty(M, \mathrm{Hom}(E, F))$ of the bundle of complex linear maps $E \to F$:

$$\begin{array}{ccccc} E & & \overset{D}{\to} & & F \\ & \searrow & & \swarrow & \\ & & M & & \end{array} .$$

If

$$\mathrm{rank}\, E - \mathrm{rank}\, F + n - 1 = 0 \tag{1}$$

then a generic section D will be injective at all but finitely many points[2]. A point $x \in M$ is called a **crossing** if $\ker\, D(x) \neq \{0\}$. A crossing is called **regular** if $\dim^c \ker\, D(x) = 1$ and

$$\mathrm{im}\, D(x) \oplus \{(\bigtriangledown_v D)(x)\zeta \,|\, v \in T_x M\} = F_x$$

for some (and hence every) nonzero vector $\zeta \in \ker\, D(x)$. For every regular crossing define $\nu(x, D) = +1$ or $\nu(x, D) = -1$ according to the orientations in the direct sum, Then we have

Theorem 1 *Assume (1) and let $D \in C^\infty(M, \mathrm{Hom}(E, F))$ be a section with only regular crossings. Then the crossing index of D is given by*

$$\nu(D) = \sum_{\ker\, D(x) \neq \{0\}} \nu(x, D) = \int_M c_n(F \ominus E).$$

II.The Topological Quantum Field theory

In this section, we will construct a cohomological quantum field theory and explain why the partition function is the crossing number. We will divide the proceedure into three case: Case 1, $E = M \times C$; Case 2, $E = M \times C^{N+1}$; Case 3, E is a nontrivial bundle.

Case 1. $E = M \times C$

Let h_{ab} be a Hermite metric and A be a $U(n)$ connection on F.

Condition (1) says that rank$F = n$ and a section of Hom(E, F) will be defined by a section s of F. A crossing x is simply a zero of s, it is regular if and only if s intersects the zero section of F transverseally at x, and the crossing index is the intersection number.

We will consider a system with a topological symmetry $Q(Q^2 = 0)$ that carries charge one with respect to a "ghost number" operator U. There will be two multiplets. The first consists of local coordinates u^i on M (with $U = 0$) together with fermions ψ^i tangent to M, with $U = 1$. The transformation laws are

$$\begin{array}{l} \delta u^i = i\epsilon\psi^i, \\ \delta\psi^i = 0, \end{array} \tag{2}$$

where ϵ is an anticommuting parameter. We also define δ_0 to be the variation with ϵ removed, for example, $\delta_0 u^i = i\psi^i$. The second multiplet consists of an anticommuting section χ^a of F with $U = -1$, and a commuting section H^a of F; H has $U = 0$. The transformation laws are

$$\begin{array}{l} \delta\chi^a = \epsilon H^a - \epsilon\delta_0 u^i A^a_{ib}\chi^b, \\ \delta H^a = \epsilon\delta_0 u^i A^a_{ib} H^b - \frac{\epsilon}{2}\delta_0 u^i \delta_0 u^j F^a_{ijb}\chi^b \end{array} \tag{3}$$

where F^a_{ijb} is the convature of the connection A. The formula is a covariant version of the more naive $\delta\chi^a = \epsilon H^a, \delta H^a = 0$.

Let $\{e_1, e_2, ..., e_n\}$ be the complex local frame of F. To construct the topological field, we first consider F as a real vector bundle F^R. Then it is an oriented vector bundle with orientation $\{e_1, ie_1, e_2, ie_2, ..., e_n, ie_n\}$. Therefore, we have

$$\begin{array}{l} \chi = \ {}_R\chi + i\ {}_I\chi, \\ H = \ {}_RH + i\ {}_IH, \\ A = \ {}_RA + i\ {}_IA, \\ F = \ {}_RF + i\ {}_IF. \end{array} \tag{4}$$

where ${}_R\chi,\ {}_I\chi,\ {}_RH,\ {}_IH,\ {}_RA,\ {}_IA,\ {}_RF,\ {}_IF$ are real fields.

Equations(2), (3), (4) imply

$$\begin{aligned}
&\delta\ {}_R\chi^a = \epsilon\ {}_RH^a - \epsilon\delta_0 u^i(\ {}_RA^a_{ib}\ {}_R\chi^b -\ {}_IA^a_{ib}\ {}_I\chi^b).\\
&\delta\ {}_I\chi^a = \epsilon\ {}_IH^a - \epsilon\delta_0 u^i(\ {}_IA^a_{ib}\ {}_R\chi^b +\ {}_IA^a_{ib}\ {}_I\chi^b),\\
&\delta\ {}_RH^a = \epsilon\delta_0 u^i(\ {}_RA^a_{ib}\ {}_RH^b -\ {}_IA^a_{ib}\ {}_IH^b) - \tfrac{\epsilon}{2}\delta_0 u^i\delta_0 u^j(\ {}_RF^a_{ijb}\ {}_R\chi^b -\ {}_IF^a_{ijb}\ {}_I\chi^b),\\
&\delta\ {}_IH^a = \epsilon\delta_0 u^i(\ {}_RA^a_{ib}\ {}_IH^b +\ {}_IA^a_{ib}\ {}_RH^b) - \tfrac{\epsilon}{2}\delta_0 u^i\delta_0 u^j(\ {}_RF^a_{ijb}\ {}_I\chi^b +\ {}_IF^a_{ijb}\ {}_R\chi^b).
\end{aligned} \tag{5}$$

Define

$$W = \frac{1}{2\lambda}(\chi, H + 2is) \tag{6}$$

with λ is a small positive real number, the fields χ, H, s are to be thinked as real fields, (,) is the Riemannian metric of the bundle F^R induced by the Hermitian metric of F.

Define the Lagrangian L to be of the form as follows:

$$L = \delta_0 W \tag{7}$$

and

$$B = \begin{pmatrix} {}_RA^a{}_b & -\ {}_IA^a{}_b \\ {}_IA^a{}_b & {}_RA^a{}_b \end{pmatrix} \tag{8}$$

$$G = \begin{pmatrix} {}_RF^a_{\ b} & -\ {}_IF^a_{\ b} \\ {}_IF^a_{\ b} & {}_RF^a_{\ b} \end{pmatrix}. \tag{9}$$

Since A is an $U(n)$ connection, we have

$$\overline{A}^t = -A \tag{10}$$

Which is equivalalent to the following form,

$$\begin{aligned} {}_RA^t &= -\ {}_RA,\\ {}_IA^t &=\ {}_IA. \end{aligned} \tag{11}$$

and

$$B^t = -B. \tag{12}$$

which imply that B is an $O(2n)$ connection.

Since

$$\begin{aligned} F =&\ dA + A\wedge A\\ =&\ d\ {}_RA + i\ {}_IA +\ {}_RA\wedge\ {}_RA -\ {}_IA\wedge\ {}_IA + i(\ {}_IA\wedge\ {}_RA +\ {}_RA\wedge\ {}_IA), \end{aligned} \tag{13}$$

we have

$$\begin{aligned} dB + B\wedge B = &\begin{pmatrix} d\ {}_RA & -d\ {}_IA \\ d\ {}_IA & d\ {}_RA \end{pmatrix}\\ + &\begin{pmatrix} {}_RA\wedge\ {}_RA -\ {}_IA\wedge\ {}_IA & -\ {}_RA\wedge\ {}_IA -\ {}_IA\wedge\ {}_RA \\ {}_IA\wedge\ {}_RA +\ {}_RA\wedge\ {}_IA & -\ {}_RA\wedge\ {}_IA -\ {}_IA\wedge\ {}_RA \end{pmatrix}. \end{aligned} \tag{14}$$

That is G is the curvature of the connection B.

Now, in view of the above relations, we obtain

$$L=\frac{1}{2\lambda}(H,H+2is)+\frac{1}{4\lambda}(\chi,G^a_{ijb}\psi^i\psi^j\chi^b)-\frac{1}{\lambda}(\chi,\nabla^B_i s\psi^i). \tag{15}$$

The partition function is defined to be

$$Z=(\frac{\lambda}{2\pi})^n\int \mathcal{D}u\mathcal{D}\psi\mathcal{D}\chi\mathcal{D}H\exp(-L). \tag{16}$$

As a first step to evaluate the above integration , we make an integration over H and obtain

$$Z=(\frac{\lambda}{2\pi})^n\int \mathcal{D}u\mathcal{D}\psi\mathcal{D}\chi\exp(-\frac{(s,s)}{2\lambda}-\frac{1}{\lambda}g_{ab}\chi^a\frac{\partial s^b}{\partial u^i}\psi^i+\frac{1}{2\lambda}G_{ijab}\psi^i\psi^j\chi^a\chi^b), \tag{17}$$

where s is transverse to the zero section of F^R. Similar to the method used in [1], we have

$$\begin{aligned} Z= & \lim_{\lambda\to 0}(\tfrac{\lambda}{2\pi})^n\int \mathcal{D}u\mathcal{D}\psi\mathcal{D}\chi\exp(-\frac{(s,s)}{2\lambda}-\frac{1}{\lambda}g_{ab}\chi^a\frac{\partial s^b}{\partial u^i}\psi^i+\frac{1}{2\lambda}G_{ijab}\psi^i\psi^j\chi^a\chi^b) \\ = & \nu(D). \end{aligned} \tag{18}$$

Because the partition function Z is independent of s, it is only necessary to consider the case $s=0$, that is

$$Z=(\frac{\lambda}{2\pi})^n\int \mathcal{D}u\mathcal{D}\psi\mathcal{D}\chi\exp(\frac{1}{2\lambda}G_{ijab}\psi^i\psi^j\chi^a\chi^b). \tag{19}$$

To evaluate the integration , we firstly choose a proper viebins such that G takes the forms

$$(G_{ijab})=\begin{pmatrix} 0 & \lambda_{ij1} & & & \\ -\lambda_{ij1} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_{ijn} \\ & & & -\lambda_{ijn} & 0 \end{pmatrix}. \tag{20}$$

So we have

$$G_{ijab}\psi^i\psi^j\chi^a\chi^b=\sum_{k=1}^n 2\lambda_{ijk}\psi^i\psi^j\chi^{2k-1}\chi^{2k}. \tag{21}$$

The partition function is reduced to

$$\begin{aligned} Z &= (\tfrac{\lambda}{2\pi})^n\int \mathcal{D}u\mathcal{D}\psi\mathcal{D}\chi\exp(-L) \\ &= \int \mathcal{D}u\mathcal{D}\psi\mathcal{D}\exp(\frac{\lambda_{ijk}\psi^i\psi^j\chi^{2k-1}\chi^{2k}}{\lambda}) \\ &= \int \mathcal{D}u(\frac{1}{2\pi})^n\prod_{k=1}^n\lambda_{ijk}. \end{aligned} \tag{22}$$

In the above viebins, we know that F is diagonal,

$$F = \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_n \end{pmatrix}. \tag{23}$$

and can easily calculate the determinant given below

$$\begin{aligned} \det(\lambda I &- \frac{i}{2\pi}\mathrm{diag}(i\lambda_1, ..., i\lambda_n)) \\ &= \det(\mathrm{diag}(\lambda + \frac{\lambda_1}{2\pi}, ..., \lambda + \frac{\lambda_n}{2\pi})) \\ &= \frac{\lambda_1 \cdots \lambda_n}{(2\pi)^n} + \cdots + \lambda^n. \end{aligned} \tag{24}$$

So we have the Chern class

$$c_n(F) = \frac{1}{(2\pi)^N}\lambda_1 \cdots \lambda_n. \tag{25}$$

and the crossing number of D

$$\begin{aligned} \nu(D) &= \int_M c_n(F) \\ &= \int_M c_n(F_E). \end{aligned} \tag{26}$$

Case 2.$E = M \times C^{N+1}$

Condition (1) says that rank$F = n + N$. Considering that D only have regular crossing, we know that there is a global frame $e_1, \cdots, e_N, e_{N+1}$ of E such that

$$\begin{aligned} s_1(x) &= D(x)e_1, \\ &\vdots \\ s_N(x) &= D(x)e_N \end{aligned} \tag{27}$$

is linearly independent.

Then $\{s_1, \cdots, s_N\}$ span a sunbundle F_0 of F, and F_0 is a trivall bundle.

Choose a veibine $\{f_1, \cdots, f_N, f_{N+1}, \cdots, f_{N+n}\}$ in bundle F such that

$$f_i(x) = s_i(x), \qquad 1 \le i \le N. \tag{28}$$

and

$$D = s_1 \oplus \cdots \oplus s_{N+1}. \tag{29}$$

In the matrix form, we have

$$D = \begin{pmatrix} I_{N\times N} & 0 \\ 0 & s_{N+1} \end{pmatrix}. \tag{30}$$

Let $\overline{s}(x) = [s_N + 1] : M \to F/F_0$. Since $D(x)$ only have regular crossing, so does $\overline{s}$, and

$$\nu(D) = \nu(\overline{s}). \tag{31}$$

Analogous to the first case, we know

$$\begin{aligned} \nu(\overline{s}) &= \int_M c_n(F/F_0 \ominus E) \\ &= \int_M c_n(F). \end{aligned} \tag{32}$$

Case 3. E is a nontrivial bundle

In the case 1 and 2, we have proved the theorem with E being the trivial bundle. However, the general case can easily be reduced to what we have disscussed. We know there exist a bundle E' such that $E \oplus E'$ is a trivial bundle. Let

$$D' = D \oplus \mathrm{id} : E \oplus E' \to F \oplus E'. \tag{33}$$

D only have regular crossing, so does D'. So we have

$$\begin{aligned} \nu(D) &= \nu(D') \\ &= \int_M c_n((F \oplus E') \ominus (E \oplus E')) \\ &= \int_M c_n(F \ominus E). \end{aligned} \tag{34}$$

This prove the theorem.

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